Uniqueness of vanishing viscosity mean curvature flow solution in two sub-Riemannian structures

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1 Introduction

Horizontal mean curvature flow describes the evolution of a surface whose points move in the normal direction, with speed equal to the curvature. The first results on this topic have been proven by Altschuler-Grayson [1], Gage [20, 22], Gage-Hamilton [21], Grayson [23] and Huisken [25], with differential geometry methods. Since mean curvature flow can develop singularities even for initially smooth surfaces, (see for example [18]), different notions of weak solution were proposed, to study the flow after singularities: Brakke introduced in [3] an approach based on the notion of varifold and geometric measure theory, Evans-Spruck [18, 15, 16, 17] and Chen-Giga-Goto's [7] independently studied existence and uniqueness of viscosity solutions via level set methods.

The level set method identifies the evolving surface at time t as level set $M_t = \{x \in \mathbb{R}^n : \ u(x,t) = 0\}$ of a function u, which is solution of a differential equation. In the Euclidean setting this equation reads:

$$\partial_t u(x,t) = |\nabla u| K = \sum_{i,j=1}^{n} \Big(\delta_{ij} - \frac{\partial_{x_i} u \, \partial_{x_j} u}{|\nabla u|^2}\Big) \partial_{x_i x_j} u. \tag{1}$$

Here $K = \operatorname{div}\Big(\dfrac{\nabla u}{|\nabla u|}\Big)$ denotes the mean curvature at the corresponding point. Note that (1) is degenerate at singularities, i.e., when $|\nabla u| = 0$.

The present paper focuses on the sub-Riemannian analogue of the mean curvature flow in two special settings.

A sub-Riemannian structure is defined by a triple (G, HG, g_0), where G is a group, HG a sub-bundle of the tangent space, and g_0 a metric on HG. We consider here a group G of topological dimension 3, which can be either the Heisenberg group or the group of rigid motion SE(2), and we denote the tangent space at every point with TG. The sub-bundle HG is generated by the vector fields X_1 and X_2 at every point, and has the bracket generating properties at step 2, i.e. X_1, X_2, and $[X_1, X_2]$ span the tangent space to G at every point. Finally we will choose a metric g_0 on HG, which will make X_1 and X_2 orthonormal. In particular while considering SE(2), the underlying manifold will be $G = R^2 \times S^1$, its elements will be expressed by $p = (x, y, \theta) \in \mathrm{SE}(2)$ so that $x,\ y$ denote the spatial components and θ the angular component. We will make the choice of vector fields

$$X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y, \quad X_2 = \partial_\theta,$$

at all $p \in G$, which satisfy the bracket generating condition, as it is easy to verify. While studying the Heisenberg group we will choose the underlying manifold as $G = \mathbb{R}^3$ with the choice of vector fields

$$X_1 = \partial_x - \frac{y}{2}\partial_\theta, \quad X_2 = \partial_y + \frac{x}{2}\partial_\theta.$$

Again $p = (x, y, \theta)$ denotes the elements of G.

We explicitly note that the Heisenberg group can be considered as the limit structure obtained from SE(2) via a blow up procedure (see [34], [50] and [60]), hence those two structures, which have completely different group laws share the same local structure. This is why they can be studied together, and they can be used as models of the same types of problems. In particular the interest in studying motion by

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curvature in these two groups comes from application of image impainting through models of the visual cortex. The first layer (V1) of mammalian visual cortex was modeled as a smooth sub-Riemannian surface with local structure of the Heisenberg group in [28], and with the SE(2) geometry in [10]. As a consequence, models of image completion inspired by the functionality of the cortex has been proposed in [26] and [10].

The horizontal gradient is defined as

$$\nabla_0 = (X_1, X_2).$$

Additionally the horizontal vector fields and gradient corresponding to an element $p \in G$ are expressed by X_1^p, X_2^p and $\nabla_0^p = (X_1^p, X_2^p)$.

If we consider a regular surface zero level set $M = \{p \in G : u(p) = 0\}$ of a smooth function u, we call the points at which the horizontal gradient vanishes characteristic points of the surface and denote the set of such points as $\Sigma(M) = \{p \in M : \; \left|\nabla_0 u(p,t)\right| = 0\}$. At non characteristic points the horizontal normal is defined as

$$\nu_0 = \frac{\nabla_0 u}{|\nabla_0 u|},$$

and the horizontal mean curvature of the manifold M is given by

$$K_0 = \sum_{i=1}^{2} X_i \nu_{0,i}.$$

As in the Euclidean setting horizontal mean curvature flow is the evolution of a surface $M_0 \subset G$, with normal speed equal to the curvature. If M_0 is the level set of a function u_0, the flow at time T will be identified as the level set $M_T = \{p \in G : \; u(p,T) = 0\}$ of the solution of the following degenerate problem:

$$\begin{cases} \partial_t u = \sum\limits_{i,j=1}^{2} \left(\delta_{ij} - \frac{X_i u X_j u}{|\nabla_0 u|^2}\right)(X_i X_j) u & \text{in } G \times (0,\infty) \\ u(.,0) = u_0(.) \quad \text{on } G \times \{0\}. \end{cases} \tag{2}$$

Immediate difficulty with (1) is that it is not defined at $p \in \Sigma(M_t)$. Furthermore surfaces which are smooth in the Euclidean sense have in general characteristic points. Hence existence of a viscosity solution is known in a very general setting, (see for example [5] and [9]), while the problem of uniqueness is still largely open. Capogna and Citti [5] proved uniqueness of evolving graphs over a Carnot group, since graphs have no characteristic points [14]. In the special case of the Heisenberg group H^1 Ferrari-Liu-Manfredi [19] provided uniqueness under the assumption of axisymmetricity of solutions to (1).

Finally note that both those studies treated (1) directly via viscosity solutions in Crandall-Ishii-Lions sense (see [11], [13] and [8] for more details).

Here we drop such aforementioned symmetry assumptions but we restrict ourselves to vanishing viscosity solutions. More precisely we introduce the regularized problem

$$\begin{cases} \partial_t u^\epsilon = \sum\limits_{i,j=1}^{2} \left(\delta_{ij} - \dfrac{X_i u^\epsilon X_j u^\epsilon}{|\nabla_0 u^\epsilon|^2 + \epsilon^2}\right)(X_i X_j) u^\epsilon & \text{in } G \times (0,\infty) \\ u^\epsilon(.,0) = u_0(.) \quad \text{on } G \times \{0\}. \end{cases} \tag{3}$$

Then we will use the following vanishing viscosity solution definition:

Definition 1. *A function u is called vanishing viscosity solution of* (1) *if for any $\varepsilon > 0$ there exists a solution u^ε of the problem* (1) *and there exists*

$$\lim_{\varepsilon \to 0} u^\varepsilon = u.$$

Our main result is the following theorem.

Theorem 1.1. *For every $\alpha \in (0, \frac{1}{2})$ and $0 < T < \infty$ there exists a constant $M = M(u_0, T, \alpha)$ such that*

$$\sup_{\xi \in SE(2), 0 \le t \le T} |(u^{\epsilon_1} - u^{\epsilon_2})(\xi, t)| \le M(\epsilon_1 - \epsilon_2)^\alpha,$$

for all $0 < \epsilon_1, \epsilon_2 < 1$ *and* $\epsilon_2 < \frac{\epsilon_1}{2}$.

From Theorem 1.1 uniqueness of vanishing viscosity solutions immediately follows

Corollary 1.1.1. *Vanishing viscosity solution u is unique.*

Note that, with the Riemannian approximation we give a formal meaning to regularized operator $|\nabla_0 u| K_0$ at characteristic points. Indeed, formally if p is characteristic for every $\epsilon > 0$ then

$$\sum_{i,j=1}^{2} \Big(\delta_{ij} - \frac{X_i u(p) X_j u(p)}{|\nabla_0 u(p)|^2 + \epsilon^2}\Big)(X_i X_j)(u(p)) = \sum_{i=1}^{2} (X_i X_i) u(p),$$

which is the Laplace operator of the function u in this setting.

The paper is organized as follows. In Section 2, we provide some basics of SE(2) and H^1 sub-Riemannian geometries. Furthermore, we provide the notions of vanishing viscosity and viscosity, and describe the generalized flow. In Section 3 we provide the proof of uniqueness of solution to horizontal mean curvature flow in SE(2) setting. In Section 4, we adapt the proof of uniqueness to H^1 setting and provide the analogous results within this setting.

2 Definitions and preliminary results

2.1 Relation between viscosity and vanishing viscosity solutions

Following [5, 11] we define viscosity solutions as:

Definition 2. *(Viscosity solution) A function $u \in C(G \times [0,\infty))$ is called a viscosity sub-solution (resp. sup-solution) to* (1) *in $G \times [0,\infty)$ if for any $(x,t) \in G \times [0,\infty)$ and any test function $\phi \in C^2(G \times [0,\infty))$ such that $u - \phi$ attains its maximum (resp. minimum) at (x,t), it satisfies*

$$\partial_t \phi(s,t) \leq \text{(resp. } \geq\text{)} \begin{cases} \sum\limits_{i,j=1}^{2} \Big(\delta_{ij} - \frac{X_i\phi X_j\phi}{|\nabla_0\phi|^2}\Big)(X_i X_j)\phi & \text{if } |\nabla_0 \phi| \neq 0, \\ \sum\limits_{i,j=1}^{2} \Big(\delta_{ij} - p_i p_j\Big)(X_i X_j)\phi & \text{for some } p \in \mathbb{R}^2,\ |p| \leq 1, \text{ if } |\nabla_0\phi| = 0. \end{cases}$$

A viscosity solution u to (1) *is a function which is both viscosity sub- and sup-solution.*

We explicitly remark that the set of viscosity solutions refer to a larger set than the set of vanishing viscosity solutions. In other words, a vanishing viscosity solution to (1) is the viscosity solution which is the limit of a solution to (1) as $\epsilon \to 0$. Consequently vanishing viscosity solutions are also viscosity solutions while vice versa is not necessarily true (see also [9]).

2.2 Choice of the distance in SE(2) and $\mathbf{H}^1$

We introduce basics of the sub-Riemannian geometry and horizontal operators. For details we refer to [6, 12, 24].

The notion of distance in H^1 has been deeply studied by [27]. They proved that the Carnot-Carathedory (CC) distance of the space is equivalent to a ball box distance defined as follows:

$$d_H(\xi,\eta) = ((d_1^2 + d_2^2)^2 + d_3^2)^{1/4},$$

where for the case of the distance from $\eta \in \mathrm{H}^1$ to $\xi \in \mathrm{H}^1$ it is computed that

$$d_1 = x_\xi - x_\eta, \quad d_2 = y_\xi - y_\eta, \quad d_3 = (\theta_\xi - \theta_\eta) + \frac{1}{2}\big((x_\xi - x_\eta)y_\eta - (y_\xi - y_\eta)x_\eta\big).$$

As recalled before the linearization of the group SE(2) can be identified with the Heisenberg group. Hence a distance in SE(2) can be obtained as follows: for every couple of points ξ, η there exist constants $d_{i,\xi}$ and $d_{i,\eta}$ such that

$$\xi = \exp(X_{i|\eta} d_{i,\eta})(\eta) \quad \eta = \exp(X_{i|\xi} d_{i,\xi})(\xi).$$

Here we have denoted $X_{i|\eta}$ the vector fields with coefficients evaluated at the point η and $X_{i|\xi}$ the vector fields with coefficients evaluated at ξ. A good estimate of the CC distance is obtained evaluating the

Heisenberg gauge function on the exponential SE(2) increments, so that natural horizontal distance is given by

$$d_{\mathrm{SE}(2),\eta}(\xi,\eta) = ((d_{1,\eta}^2 + d_{2,\eta}^2)^2 + d_{3,\eta}^2)^{1/4} \quad \text{and} \quad d_{\mathrm{SE}(2),\xi}(\xi,\eta) = ((d_{1,\xi}^2 + d_{2,\xi}^2)^2 + d_{3,\xi}^2)^{1/4},$$

For SE(2) the coefficients $d_{i,\xi}$ and $d_{i,\eta}$ can be explicitly evaluated as

$$\begin{aligned}
d_{1,\eta} =& \cos(\theta_\eta)(x_\xi - x_\eta) + \sin(\theta_\eta)(y_\xi - y_\eta), && d_{1,\xi} = \cos(\theta_\xi)(x_\xi - x_\eta) + \sin(\theta_\xi)(y_\xi - y_\eta),\\
d_{2,\eta} =& -\sin(\theta_\eta)(x_\xi - x_\eta) + \cos(\theta_\eta)(y_\xi - y_\eta), && d_{2,\xi} = -\sin(\theta_\xi)(x_\xi - x_\eta) + \cos(\theta_\xi)(y_\xi - y_\eta),\\
d_{3,\eta} =& d_{3,\xi} = \theta_\xi - \theta_\eta.
\end{aligned}$$

Note that we can find some constants $K_1, K_2 > 0$ such that

$$K_1 d_E^{1/2} \leq d_{\mathrm{SE}(2),\xi}, d_{\mathrm{SE}(2),\eta}, d_H \leq K_2 d_E,$$

where $d_E(\xi,\eta) = |\xi - \eta| = \big((x_\xi - x_\eta)^2 + (y_\xi - y_\eta)^2 + (\theta_\xi - \theta_\eta)^2\big)^{1/2}$.

2.3 Existence and geometrical properties of the solutions

In [18] the existence of vanishing viscosity solution was established under the assumption of that the initial condition is identically 1 at infinity. The same theorem is already known in the two groups considered here: $G = \mathrm{SE}(2)$ and $G = H^1$ (see [5] for the Heisenberg group and [9] for SE(2)).

Theorem 2.1. *Assume that the initial datum u_0 is of class $C_E^1(G)$ (i.e., in the Euclidean sense) and there is a sphere of radius R such that u_0 is identically constant out of this sphere. Denoting ∇_E the standard Euclidean gradient there is a constant $\tilde{C}$ such that*

$$\tilde{C} \geq \max(\|u_0\|_{L^\infty(G)}, \|\nabla_E u_0\|_{L^\infty(G)})$$

and for every ϵ the solution of problem (1) satisfies

$$\begin{aligned}
\|u^\varepsilon(.,t)\|_{L^\infty(G)} &\leq \tilde{C},\\
\|\nabla_E u^\varepsilon(.,t)\|_{L^\infty(G)} &\leq \tilde{C}.
\end{aligned}$$

As a consequence, there exists a Lipschitz continous vanishing viscosity solution u of problem (1) which satisfies

$$\begin{aligned}
\|u(.,t)\|_{L^\infty(G)} &\leq \tilde{C}, \qquad (4)\\
\|\nabla_E u(.,t)\|_{L^\infty(G)} &\leq \tilde{C}.
\end{aligned}$$

We will prove the following theorem:

Theorem 2.2. *Assume that $u_0 \in C^\infty(G)$ and it is constant at the exterior of a sphere of radius R. Then there exists R_1 such that*

$$u(p,t) \text{ is constant for all } p \in G \text{ satisfying } |p| \geq R_1 \text{ and for all } t > 0.$$

3 Uniqueness in SE(2)

3.1 Some geometrical properties of solutions

We first need to establish in the SE(2) setting some geometrical properties of solutions of equations (1) and (1) whose Euclidean analogue has been proven by Evans [18].

Theorem 3.1. *Assume that $u_0 \in C^\infty(SE(2))$ and it is constant at the exterior of a cylinder. More precisely there exists a constant $K > 0$ such that*

$$u_0(p) \text{ is constant for all } p = (x,y,\theta) \in SE(2) \text{ satisfying } |x|^2 + |y|^2 \geq K\ .$$

Then

$$u(p,t) \text{ is constant for all } p \in SE(2) \text{ satisfying } |x|^2+|y|^2 \geq R \text{ and for all } t>0,$$

where $R>0$ is a finite number dependent only on K.

Proof. Up to a rescaling we may assume

$$|u_0| \leq 1 \text{ on SE}(2), \qquad u_0 = 0 \text{ if } \frac{|x|^2+|y|^2}{2} \geq 1.$$

Consider now the auxiliary function inspired from [4, p.5] and given by

$$v^\varepsilon(p,t) = \Psi((|x|^2+|y|^2)/2 + t\varepsilon) \quad \text{for all } p \in \text{SE}(2),\ t>0,$$

with $0<\varepsilon<1$ and

$$\Psi(s) \equiv \begin{cases} 0 & (s \geq 2), \\ (s-2)^3 & (0 \leq s \leq 2). \end{cases}$$

Clearly $\Psi \in \mathcal{C}^2\big([0,\infty)\big)$,

$$\Psi'(s) = \begin{cases} 0 & (s \geq 2), \\ 3(s-2)^2 & (0 \leq s \leq 2), \end{cases} \qquad \text{and} \qquad \Psi''(s) = \begin{cases} 0 & (s \geq 2), \\ 6(s-2) & (0 \leq s \leq 2). \end{cases}$$

From this explicit expression it immediately follows that $\Psi' \geq 0$, $\Psi'' \leq 0$ and $\Psi \leq 0$, $|\Psi''| \leq 2\sqrt{3}\left(\Psi'\right)^{1/2}$, $|\Psi'|, |\Psi''| \leq 12$, for all $s>0$.

Let us evaluate (1) on the function v^ε. Then we have

$$\begin{aligned} v_t^\varepsilon - \sum_{i,j=1}^{2} \left(\delta_{ij} - \frac{X_i v^\varepsilon X_j v^\varepsilon}{\varepsilon^2+|\nabla_0 v^\varepsilon|^2}\right)(X_iX_j)v^\varepsilon = \\ = \varepsilon\Psi' - \frac{\big(\varepsilon^2+(X_2v^\varepsilon)^2\big)(X_1X_1)v^\varepsilon}{\varepsilon^2+|\nabla_0 v^\varepsilon|^2} - \frac{\big(\varepsilon^2+(X_1v^\varepsilon)^2\big)(X_2X_2)v^\varepsilon}{\varepsilon^2+|\nabla_0 v^\varepsilon|^2} + 2\frac{X_1v^\varepsilon X_2 v^\varepsilon (X_1X_2)v^\varepsilon}{\varepsilon^2+|\nabla_0 v^\varepsilon|^2}, \end{aligned} \tag{5}$$

and compute the first and second order derivatives of v^ε explicitly as

$$X_2 v^\varepsilon(p,t) = 0, \quad (X_2X_2)v^\varepsilon(p,t) = 0, \quad (X_1X_1)v^\varepsilon(p,t) = \Psi' + \Psi''(\cos(\theta)x+\sin(\theta)y)^2. \tag{6}$$

Using (3.1) and (3.1) we deduce that

$$\begin{aligned} v_t^\varepsilon - \sum_{i,j=1}^{2}\left(\delta_{ij} - \frac{X_iv^\varepsilon X_jv^\varepsilon}{\varepsilon^2+|\nabla_0 v^\varepsilon|^2}\right)(X_iX_j)v^\varepsilon &= \varepsilon\Psi' - \frac{\varepsilon^2\big(\Psi'+\Psi''(\cos(\theta)x+\sin(\theta)y)^2\big)}{\varepsilon^2+|X_1v^\varepsilon|^2} \\ &\leq \varepsilon\Psi' + \frac{\varepsilon^2 C\big(\Psi'\big)^{1/2}(\cos(\theta)x+\sin(\theta)y)^2}{\varepsilon^2+(\Psi')^2(\cos(\theta)x+\sin(\theta)y)^2}, \end{aligned} \tag{7}$$

where we use the fact that $\Psi' \geq 0$ and $|\Psi''| \leq C(\Psi')^{1/2}$. Here note that if $\varepsilon \leq |\Psi'|$ then

$$\frac{\varepsilon^2 C\big(\Psi'\big)^{1/2}(\cos(\theta)x+\sin(\theta)y)^2}{\varepsilon^2+(\Psi')^2(\cos(\theta)x+\sin(\theta)y)^2} \leq \frac{\varepsilon^2 C}{(\Psi')^{3/2}} \leq C\,\varepsilon^{1/2}. \tag{8}$$

On the other hand if $\varepsilon > |\Psi'|$ then

$$\frac{\varepsilon^2 C\big(\Psi'\big)^{1/2}(\cos(\theta)x+\sin(\theta)y)^2}{\varepsilon^2+(\Psi')^2(\cos(\theta)x+\sin(\theta)y)^2} \leq \frac{\varepsilon^{5/2}C(\cos(\theta)x+\sin(\theta)y)^2}{\varepsilon^2} \leq C\varepsilon^{1/2}, \tag{9}$$

where we considered only the case with $\big(|x|^2+|y|^2\big)/2 < 2$ since otherwise the estimate follows trivially due to that $\Psi'=0$. Employing (3.1) and (3.1), we now obtain from (3.1) that there exists a constant

$\bar{C} > 0$ such that

$$v_t^\varepsilon - \sum_{i,j=1}^{2} \left(\delta_{ij} - \frac{X_i v^\varepsilon X_j v^\varepsilon}{\varepsilon^2 + |\nabla_0 v^\varepsilon|^2}\right)(X_i X_j) v^\varepsilon \leq \bar{C}\varepsilon^{1/2}. \tag{10}$$

Now let us define

$$V^\varepsilon(p,t) := v^\varepsilon(p,t) - t\bar{C}\varepsilon^{1/2},$$

where $\bar{C}$ is the constant in (3.1). Notice that V^ε satisfies

$$V_t^\varepsilon - \sum_{i,j=1}^{2} \left(\delta_{ij} - \frac{X_i V^\varepsilon X_j V^\varepsilon}{\varepsilon^2 + |\nabla_0 V^\varepsilon|^2}\right)(X_i X_j) V^\varepsilon \leq 0,$$

and at the initial time $t = 0$

$$\begin{aligned} V^\varepsilon(p,0) =& \Psi((|x|^2 + |y|^2)/2) = 0 \quad \text{if} \quad (|x|^2 + |y|^2)/2 \geq 2, \\ -1 \leq V^\varepsilon(p,0) =& \Psi((|x|^2 + |y|^2)/2) \leq 0 \quad \text{if} \quad 1 \leq (|x|^2 + |y|^2)/2 \leq 2, \\ V^\varepsilon(p,0) =& \Psi((|x|^2 + |y|^2)/2) \leq -1 \quad \text{if} \quad 0 \leq (|x|^2 + |y|^2)/2 \leq 1, \end{aligned}$$

i.e., $V^\varepsilon(p,0) \leq u_0(p)$ for all $p \in \mathrm{SE}(2)$.

Applying the comparison principle to the regularized mean curvature equation we deduce that $V^\varepsilon \leq u^\varepsilon$ in $\mathrm{SE}(2) \times [0,\infty)$ for each $0 < \varepsilon < 1$. This result implies

$$\lim_{\varepsilon \to 0} V^\varepsilon = \Psi\Big(\frac{|x|^2 + |y|^2}{2}\Big) = 0 \leq u(p,t), \tag{11}$$

for all $t \geq 0$ and $x, y \in \mathrm{SE}(2)$ satisfying $(|x|^2 + |y|^2)/2 \geq 2$. Hence $u \geq 0$ if $(|x|^2 + |y|^2)/2 \geq 2$. Arguing in the same way with the function $\tilde{V}^\varepsilon = -V^\varepsilon$, we deduce

$$u \leq 0 \quad \text{if } (|x|^2 + |y|^2)/2 \geq 2. \tag{12}$$

Hence (3.1) and (3.1) give $u(p,t) = 0$ for all $t \in [0,T]$ and $p = (x,y,\theta)$ such that $(|x|^2 + |y|^2)/2 \geq 2$. □

Note that Theorem 3.1 guarantees that once level sets of u_0 are confined in some bounded interior region then the level sets (in particular zero level set) of vanishing viscosity solution u stay within the same region during the whole flow.

We need also the following theorem:

Theorem 3.2. *For each $\varepsilon > 0$ there exist some finite numbers $B, b > 0$ independent of ϵ such that*

$$\left|1 - u^\varepsilon(p,t)\right| \leq Be^{-b(x^2+y^2)} \quad \textit{for all } (p,t) \in SE(2) \times [0,T],$$

where T is some positive finite number denoting the final time.

Proof. As in the previous theorem we may assume that

$$|u_0| \leq 1 \text{ on } \mathrm{SE}(2), \qquad u_0 = 0 \text{ in } \frac{|x|^2 + |y|^2}{2} \geq 1. \tag{13}$$

We will estimate the function $1 - u^\varepsilon$ in terms of the exponential function $\Psi(s) = \hat{c}\mathrm{e}^{-\sigma(2T-\alpha t)s}$ and $v^\epsilon(p,t) = \Psi((x^2 + y^2))$ where $0 < \alpha < 1$, $0 < \sigma < \infty$ and $\hat{c} = 2\mathrm{e}^{4\sigma T}$.

Now we proceed by using a procedure similar to the one in the proof of [2, Theorem 5.1]. Using the

derivatives of v^ε computed in (3.1) the curvature operator on the function v^ε becomes:

$$
\begin{aligned}
v_t^\varepsilon - \sum_{i,j=1}^{2}\left(\delta_{ij} - \frac{X_i v^\varepsilon X_j v^\varepsilon}{\varepsilon^2 + |X_j v^\varepsilon|^2}\right)(X_i X_j)v^\varepsilon &= v_t^\varepsilon - \frac{\varepsilon^2\big(\Psi' + \Psi''(\cos(\theta)x + \sin(\theta)y)^2\big)}{\varepsilon^2 + |X_1 v^\varepsilon|^2} \\
&\geq \Big(\hat{c}\,\sigma\,\alpha\,(x^2+y^2)\; - 4\,\hat{c}\,\sigma^2(2T-\alpha t)^2\Big(x\cos(\theta) + y\sin(\theta)\Big)^2\Big)\mathrm{e}^{-\sigma(2T-\alpha t)(x^2+y^2)} \\
&\geq \Big(\alpha\; - 4\;\sigma(2T)^2\Big)\hat{c}\,\sigma\,(x^2+y^2)\mathrm{e}^{-\sigma(2T-\alpha t)(x^2+y^2)} \geq 0,
\end{aligned}
$$

for a suitably chosen α and σ satisfying $\alpha - 16\sigma T^2 \geq 0$. Note that

$$
(1-u^\varepsilon)_t - \sum_{i,j=1}^{2}\left(\delta_{ij} - \frac{X_i(1-u^\varepsilon)X_j(1-u^\varepsilon)}{\varepsilon^2 + |\nabla_0(1-u^\varepsilon)|^2}\right)(X_i X_j)(1-u^\varepsilon) = 0.
$$

Recall that $0 \leq |1-u_0| \leq 2$ due to (3.1). Hence, applying comparison principle to the functions $1-u^\varepsilon$ (as well as $u^\varepsilon - 1$) and v^ε we get

$$
|1-u^\varepsilon| \leq v^\varepsilon \quad \text{in } \mathrm{SE}(2)\times[0,T].
$$

The proof is complete. □

3.2 Proof of uniqueness result

Now we can prove Theorem 1.1.

Proof. Let us fix $\alpha \in (0, \frac{1}{2})$ and assume $0 < T < \infty$. We claim that there exists a constant $M \geq 0$ such that

$$
\sup_{\xi\in \mathrm{SE}(2), 0\leq t\leq T} |(u^{\epsilon_1} - u^{\epsilon_2})(\xi,t)| \leq M(\epsilon_1 - \epsilon_2)^\alpha,
$$

for all parameters $0 < \epsilon_1, \epsilon_2 < 1$ and $\epsilon_2 < \frac{\epsilon_1}{2}$.

We argue by contradiction and assume that

$$
\begin{aligned}
&\text{For each } M \geq 0 \text{ there exist } 0 < \epsilon_1 = \epsilon_1(M), \epsilon_2 = \epsilon_2(M) < 1 \text{ such that} \\
&\sup_{\xi\in \mathrm{SE}(2), 0\leq t\leq T} |(u^{\epsilon_1} - u^{\epsilon_2})(\xi,t)| > M(\epsilon_1-\epsilon_2)^\alpha > M\Big(\frac{\epsilon_1}{2}\Big)^\alpha, \\
&\text{and } \epsilon_2 < \frac{\epsilon_1}{2}.
\end{aligned}
\tag{14}
$$

Now we choose a paramenter γ such that

$$
\gamma \geq \frac{(\alpha+7)}{3}. \tag{15}
$$

For every point $\xi = (x_\xi, y_\xi, \theta)$, we will denote $\bar{\xi} = (x_\xi, y_\xi)$ and we will define

$$
\omega(\bar{\xi}, \bar{\eta}, \theta, t) = u^{\epsilon_1}(\bar{\xi}, \theta, t) - u^{\epsilon_2}(\bar{\eta}, \theta, t) - \phi(\xi, \eta, t),
$$

with

$$
\phi(\xi,\eta,t) = \frac{\mu}{\gamma}(\epsilon_1-\epsilon_2)^{1-\frac{\gamma}{2}}|\xi-\eta|_0^\gamma + \frac{M}{2T}(\epsilon_1-\epsilon_2)^\alpha t,
$$

and

$$
|\xi - \eta|_0 = |(x_\xi, y_\xi, \theta) - (x_\eta, y_\eta, \theta)|_0 = |\bar{\xi} - \bar{\eta}|_E = (|x_\xi - x_\eta|^2 - |y_\xi - y_\eta|^2)^{\frac{1}{2}}. \tag{16}
$$

We employ (3.2) and observe that

$$\sup_{\substack{\bar\xi,\bar\eta\in R^2,\\ 0\le\theta\le 2\pi, 0\le t\le T}} \omega(\bar\xi,\bar\eta,\theta,t) \ge \sup_{\xi\in\mathrm{SE}(2), 0\le t\le T} \omega(\xi,\xi,t) \ge \sup_{\xi\in\mathrm{SE}(2)} \omega(\xi,\xi,T)$$
$$\ge M(\epsilon_1-\epsilon_2)^\alpha - \frac{M}{2T}(\epsilon_1-\epsilon_2)^\alpha T = \frac{M}{2}(\epsilon_1-\epsilon_2)^\alpha$$
$$\ge \frac{M}{2}\Big(\frac{\epsilon_1}{2}\Big)^\alpha.$$

We proceed in two steps. We first show that $\sup\omega$ is attained at a point within some interior bounded region. In the second step we reach a contradiction, evaluating the equation at the maximum point.

Step 1

In this first step we prove that, under the contradictory assumption (3.2), the function ω must attain a maximum.

As a result of (2.1) we have

$$\omega(\bar\xi,\bar\eta,\theta,t) \le \sup_{\xi\in\mathrm{SE}(2),0\le t\le T} u^{\epsilon_1}(\xi,t) + \sup_{\eta\in\mathrm{SE}(2),0\le t\le T}\Big(-u^{\epsilon_2}(\eta,t)\Big) - \frac{\mu}{\gamma}(\epsilon_1-\epsilon_2)^{1-\frac{\gamma}{2}}|\xi-\eta|_0^\gamma$$
$$\le 2\tilde C - \frac{\mu}{\gamma}(\epsilon_1-\epsilon_2)^{1-\frac{\gamma}{2}}|\xi-\eta|_0^\gamma.$$

This assertion implies that

$$\omega(\bar\xi,\bar\eta,\theta,t) \le \frac{M}{4}(\epsilon_1-\epsilon_2)^\alpha,$$

when

$$|\xi-\eta|_0 \ge r := \Big(\frac{2\gamma\tilde C}{\mu}(\epsilon_1-\epsilon_2)^{\frac{\gamma}{2}-1} - \frac{M\gamma}{4\mu}(\epsilon_1-\epsilon_2)^{\alpha+\frac{\gamma}{2}-1}\Big)^{\frac{1}{\gamma}}.$$

So by (3.2) we deduce that

$$\sup_{\substack{\bar\xi,\bar\eta\in R^2,\\ \theta\in[0,2\pi]\ 0\le t\le T}} \omega(\bar\xi,\bar\eta,\theta,t) = \sup_{\substack{\bar\xi,\bar\eta\ \in\ R^2 \text{ with } |\xi-\eta|_0\le r,\\ 0\le t\le T}} \omega(\bar\xi,\bar\eta,\theta,t).$$

Employing Theorem 3.2 we find that

$$\omega(\bar\xi,\bar\eta,\theta,t) \le \big|1-u^{\epsilon_1}(\bar\xi,\theta,t)\big| + \big|1-u^{\epsilon_2}(\bar\eta,\theta,t)\big|$$
$$\le B_{\epsilon_1}\mathrm{e}^{-b_{\epsilon_1}|\xi|_0^2} + B_{\epsilon_2}\mathrm{e}^{-b_{\epsilon_2}|\eta|_0^2} \le \frac{B_{\epsilon_1}}{b_{\epsilon_1}}\frac{1}{|\xi|_0^2} + \frac{B_{\epsilon_2}}{b_{\epsilon_2}}\frac{1}{|\eta|_0^2}, \tag{17}$$

for $|\xi|_0, |\eta|_0 \ne 0$. By the triangular inequality we find that

$$|\eta|_0 \ge |\xi|_0 - r.$$

For $|\xi|_0 \ge \tilde R_\xi := r + \sqrt{\frac{4B_{\epsilon_2}}{Mb_{\epsilon_2}}(\epsilon_1-\epsilon_2)^{-\alpha}}$, we see via Theorem 3.2 that

$$|1-u^{\epsilon_2}(\bar\eta,\theta,t)| \le B_{\epsilon_2}\mathrm{e}^{-b_{\epsilon_2}|\eta|_0^2} \le \frac{B_{\epsilon_2}}{b_{\epsilon_2}}\frac{1}{|\eta|_0^2} \le \frac{B_{\epsilon_2}}{b_{\epsilon_2}}\frac{1}{\big(|\xi|_0-r\big)^2}$$
$$\le \frac{B_{\epsilon_2}}{b_{\epsilon_2}}\frac{1}{\big(\tilde R_\xi - r\big)^2} = \frac{M}{4}(\epsilon_1-\epsilon_2)^\alpha. \tag{18}$$

The same reasoning gives also

$$|1-u^{\epsilon_1}(\bar\xi,\theta,t)| \le \frac{M}{4}(\epsilon_1-\epsilon_2)^\alpha, \tag{19}$$

for $|\xi|_0 \geq \tilde{R}_\xi$ and $|\eta-\xi|_0 \leq r$. We plug (3.2) and (3.2) in (3.2) and find

$$\omega(\bar{\xi},\bar{\eta},\theta,t) \leq \frac{B_{\epsilon_1}}{b_{\epsilon_1}}\frac{1}{|\xi|_0^2} + \frac{B_{\epsilon_2}}{b_{\epsilon_2}}\frac{1}{|\eta|_0^2} \leq \frac{M}{2}(\epsilon_1-\epsilon_2)^\alpha.$$

Therefore we deduce from (3.2) that

$$\sup_{\substack{\bar{\xi},\bar{\eta}\in R^2,\\ \theta\in[0,\pi]\ 0\leq t\leq T}} \omega(\bar{\xi},\bar{\eta},\theta,t) = \sup_{\substack{\bar{\xi},\bar{\eta}\in R^2,\ \theta\in[0,\pi]\ ;\\ |\xi-\eta|_0\leq r,\ |\xi|_0\leq \tilde{R}_\xi,\ 0\leq t\leq T}} \omega(\bar{\xi},\bar{\eta},\theta,t) = \omega(\hat{\xi},\hat{\eta},\hat{\theta},\hat{t}),$$

Finally we see that $\hat{t}>0$ must hold for a proper selection of μ. Observe for $t=0$ that

$$\begin{aligned}\omega(\bar{\xi},\bar{\eta},\theta,0) &= u_0(\xi,\theta)-u_0(\eta,\theta)-\frac{\mu}{\gamma}(\epsilon_1-\epsilon_2)^{1-\frac{\gamma}{2}}|\xi-\eta|_0^\gamma \\ &\leq |\xi-\eta|_0\Big(\mathrm{Lip}(u_0)-\frac{\mu}{\gamma}\epsilon^{1-\frac{\gamma}{2}}|\xi-\eta|_0^{\gamma-1}\Big),\end{aligned} \tag{20}$$

with $\mathrm{Lip}(u_0)$ denoting Lipschitz constant of u_0.

Consider the first case with $|\xi-\eta|_0 \leq \frac{1}{4\mathrm{Lip}(u_0)}M\epsilon^\alpha$. Then (3.2) can be written as

$$\omega(\bar{\xi},\bar{\eta},\theta,0) \leq \frac{M}{4}\epsilon^\alpha,$$

which together with (3.2) ensures that the maximum is not taken for $t=0$.

Now consider the second case where $|\xi-\eta|_0 > \frac{1}{4\mathrm{Lip}(u_0)}M\epsilon^\alpha$. Then for

$$\mu = \frac{\gamma 4^\gamma \mathrm{Lip}(u_0)^\gamma}{M^{\gamma-1}}, \tag{21}$$

observe that (3.2), (3.2) and (3.2) give

$$\omega(\bar{\xi},\bar{\eta},\theta,0) \leq |\xi-\eta|_0\Big(\mathrm{Lip}(u_0)-\frac{\mu}{\gamma}\epsilon^{1-\frac{\gamma}{2}}|\xi-\eta|_0^{\gamma-1}\Big) \leq 0,$$

and applying again (3.2) we obtain that $\hat{t}>0$.

Step 2

We will prove that if ω attains a maximum, we reach a contradiction showing that (3.2) cannot hold.

Since $\sup\omega$ is attained at an interior point $(\hat{\xi},\hat{\eta},\hat{\theta},\hat{t})$ and $u^{\epsilon_1},u^{\epsilon_2}\in C^\infty$ at the maximum point we have $X_i^\xi u^{\epsilon_1}(\hat{\xi},\hat{\theta},\hat{t}) = X_i^\xi\phi(\hat{\xi},\hat{\eta},\hat{t})$ and $-X_i^\eta u^{\epsilon_2}(\hat{\eta},\hat{\theta},\hat{t}) = X_i^\eta\phi(\hat{\xi},\hat{\eta},\hat{t})$ (note that ϕ is independent of θ due to (3.2)) and

$$\begin{aligned}&\Big(\delta_{ij}-\frac{X_i^\xi u^{\epsilon_1}(\hat{\xi},\hat{\theta},\hat{t})X_j^\xi u^{\epsilon_1}(\hat{\xi},\hat{\theta},\hat{t})}{\epsilon_1^2+|\nabla_0^\xi u^{\epsilon_1}(\hat{\xi},\hat{\theta},\hat{t})|^2}\Big)\Big((X_i^\xi X_j^\xi)u^{\epsilon_1}(\hat{\xi},\hat{\theta},\hat{t})-(X_i^\xi X_j^\xi)\phi(\hat{\xi},\hat{\eta},\hat{t})\Big)\leq 0,\\ &\Big(\delta_{ij}-\frac{X_i^\eta u^{\epsilon_2}(\hat{\eta},\hat{\theta},\hat{t})X_j^\eta u^{\epsilon_2}(\hat{\eta},\hat{\theta},\hat{t})}{\epsilon_2^2+|\nabla_0^\eta u^{\epsilon_2}(\hat{\eta},\hat{\theta},\hat{t})|^2}\Big)\Big(-(X_i^\eta X_j^\eta)u^{\epsilon_2}(\hat{\eta},\hat{\theta},\hat{t})-(X_i^\eta X_j^\eta)\phi(\hat{\xi},\hat{\eta},\hat{t})\Big)\leq 0.\end{aligned}$$

Therefore we may write

$$\frac{M\epsilon^\alpha}{2T} = \partial_t\phi(\hat{\xi},\hat{\eta},\hat{t}) = \partial_t u^{\epsilon_1}(\hat{\xi},\hat{t})-\partial_t u^{\epsilon_2}(\hat{\eta},\hat{t}) =$$

$$\begin{aligned}=\sum_{i,j=1}^2 &\Big(\delta_{ij}-\frac{X_i^\xi u^{\epsilon_1}(\hat{\xi},\hat{\theta},\hat{t})X_j^\xi u^{\epsilon_1}(\hat{\xi},\hat{\theta},\hat{t})}{\epsilon_1^2+|\nabla_0^\xi u^{\epsilon_1}(\hat{\xi},\hat{\theta},\hat{t})|^2}\Big)(X_i^\xi X_j^\xi)u^{\epsilon_1}(\hat{\xi},\hat{\theta},\hat{t})-\\ \sum_{i,j=1}^2 &\Big(\delta_{ij}-\frac{X_i^\eta u^{\epsilon_2}(\hat{\eta},\hat{\theta},\hat{t})X_j^\eta u^{\epsilon_2}(\hat{\eta},\hat{\theta},\hat{t})}{\epsilon_2^2+|\nabla_0^\eta u^{\epsilon_2}(\hat{\eta},\hat{\theta},\hat{t})|^2}\Big)(X_i^\eta X_j^\eta)u^{\epsilon_2}(\hat{\eta},\hat{\theta},\hat{t}) \leq\end{aligned}$$

$$\leq \sum_{i,j=1}^{2}\Big(\delta_{ij}-\frac{X_i^\xi\phi(\hat\xi,\hat\theta,\hat t)X_j^\xi\phi(\hat\xi,\hat\theta,\hat t)}{\epsilon_1^2+|\nabla_0^\xi\phi(\hat\xi,\hat\theta,\hat t)|^2}\Big)(X_i^\xi X_j^\xi)\phi(\hat\xi,\hat\theta,\hat t)-$$

$$\sum_{i,j=1}^{2}\Big(\delta_{ij}-\frac{X_i^\eta\phi(\hat\eta,\hat\theta,\hat t)X_j^\eta\phi(\hat\eta,\hat\theta,\hat t)}{\epsilon_2^2+|\nabla_0^\eta\phi(\hat\eta,\hat\theta,\hat t)|^2}\Big)(X_i^\eta X_j^\eta)\phi(\hat\eta,\hat\theta,\hat t).$$

Since $X_2^\xi\phi = X_2^\eta\phi = 0$, $(X_1^\xi + X_1^\eta)\phi = 0$, it follows that

$$\frac{M\epsilon^\alpha}{2T}\leq \epsilon_1^2\frac{X_1^\xi\phi X_1^\xi\phi}{(\epsilon_1^2+|X_1^\xi\phi|^2)^2}(X_1^\xi X_1^\xi)\phi=\left(\epsilon_1\frac{X_1^\xi\phi\sqrt{(X_1^\xi X_1^\xi)\phi}}{\epsilon_1^2+|X_1^\xi\phi|^2}\right)^2.$$

Note that for $\zeta\in\{\xi,\eta\}$ we have

$$X_1^\zeta\phi=\mu\epsilon^{-(\gamma-2)/2}|\xi-\eta|^{\gamma-2}(\cos(\theta)(x_\xi-x_\eta)+\sin(\theta)(y_\xi-y_\eta)),$$

$$|X_1^\zeta X_1^\zeta\phi|\leq\mu\epsilon^{-(\gamma-2)/2}|\xi-\eta|^{\gamma-2}.$$

Now we choose $q=\frac{4\gamma}{3\gamma-4}$, and call q' the conjugate exponent. We will consider large M values so $\mu<1$. Note that since $\gamma<4$, then $q>2$ and

$$\epsilon_1(X_1^\xi\phi)\sqrt{(X_1^\xi X_1^\xi)\phi}\leq$$

$$\epsilon^{\frac{(\gamma-2)}{q}-\frac{3(\gamma-2)}{4}+1-\frac{2}{q'}}\epsilon_1^{\frac{2}{q'}}|\xi-\eta|^{3(\gamma-2)/2}|\cos(\theta)(x_\xi-x_\eta)+\sin(\theta)(y_\xi-y_\eta)|\epsilon^{-\frac{(\gamma-2)}{q}}.$$

Using the fact that $\frac{\gamma-2}{q}-\frac{3(\gamma-2)}{4}+1-\frac{2}{q'}=\frac{3\gamma-7}{2}$ and Hölder inequality

$$\epsilon^{\frac{3\gamma-7}{2}}\Big(\epsilon_1^2+|\xi-\eta|^{3q(\gamma-2)/2}|\cos(\theta)(x_\xi-x_\eta)+\sin(\theta)(y_\xi-y_\eta)|^q\epsilon^{-(\gamma-2)}\Big)\leq$$

$$\epsilon^{\frac{3\gamma-7}{2}}\Big(\epsilon_1^2+|\xi-\eta|^{3q(\gamma-2)/2+q-2}|\cos(\theta)(x_\xi-x_\eta)+\sin(\theta)(y_\xi-y_\eta)|^2\epsilon^{-(\gamma-2)}\Big)\leq$$

(since $3q(\gamma-2)/2+q-2=q(\frac{3}{2}\gamma-2)-2=q\frac{3\gamma-4}{2}-2=2(\gamma-1)$)

$$\leq\epsilon^{\frac{3\gamma-7}{2}}\Big(\epsilon_1^2+|\xi-\eta|^{2(\gamma-1)}|\cos(\theta)(x_\xi-x_\eta)+\sin(\theta)(y_\xi-y_\eta)|^2\epsilon^{-(\gamma-2)}\Big)\leq$$

$$\leq\epsilon^{\frac{3\gamma-7}{2}}\Big(\epsilon_1^2+|X_1^\xi\phi|^2\Big).$$

Inserting this in the previous expression we find

$$\frac{M\epsilon^\alpha}{2T}\leq\left(\epsilon_1\frac{X_1^\xi\phi\sqrt{(X_1^\xi X_1^\xi)\phi}}{\epsilon_1^2+|X_1^\xi\phi|^2}\right)^2\leq\epsilon^{3\gamma-7}\leq\epsilon^\alpha,$$

since $3\gamma-7\geq\alpha$. This contradiction for sufficiently large M values proves the result.

□

4 Uniqueness in Heisenberg group

In this section we prove Theorem 1.1 in the Heisenberg group setting. We follow an analogous procedure: First we provide the geometrical properties of solution by proving Theorem 3.1 and Theorem 3.2 in the H^1 setting. Then we show that the solution is unique.

Similarly to the case of SE(2) we have $\xi=(x_\xi,y_\xi,\theta)\in H^1$ and $\eta=(x_\eta,y_\eta,\theta)\in H^1$ where $\theta\in\mathbb{R}$.

Now we prove the following theorem:

Theorem 4.1. *Assume that $u_0\in C^\infty(H^1)$ and there exists a constant $K>0$ such that*

$$u_0(p)\ \textit{is constant for all}\ p=(x,y,\theta)\in H^1\ \textit{satisfying}\ |x|^2+|y|^2\geq K\ .$$

Then

$$u(p,t) \text{ is constant for all } p \in H^1 \text{ satisfying } |x|^2+|y|^2 \geq R \text{ and for all } t>0,$$

where $R>0$ is a finite number dependent only on K.

Proof. Up to a rescaling we may assume

$$|u_0| \leq 1 \text{ on } H^1, \qquad u_0 = 0 \quad \text{if } \frac{|x|^2+|y|^2}{2} \geq 1.$$

Consider now the auxiliary function inspired from [4, p.5] and given by

$$v^\varepsilon(p,t) = \Psi((|x|^2+|y|^2)/2 + t\varepsilon) \quad \text{for all} \quad p \in H^1,\ t>0,$$

with $0<\varepsilon<1$ and

$$\Psi(s) \equiv \begin{cases} 0 & (s \geq 2), \\ (s-2)^3 & (0 \leq s \leq 2). \end{cases}$$

Clearly $\Psi \in \mathcal{C}^2\big([0,\infty)\big)$,

$$\Psi'(s) = \begin{cases} 0 & (s \geq 2), \\ 3(s-2)^2 & (0 \leq s \leq 2), \end{cases} \qquad \text{and} \quad \Psi''(s) = \begin{cases} 0 & (s \geq 2), \\ 6(s-2) & (0 \leq s \leq 2). \end{cases}$$

From this explicit expression it immediately follows that $\Psi' \geq 0$, $\Psi'' \leq 0$ and $\Psi \leq 0$, $|\Psi''| \leq 2\sqrt{3}\left(\Psi'\right)^{1/2}$, $|\Psi'|, |\Psi''| \leq 12$, for all $s>0$.

We compute the first and second order derivatives of v^ε explicitly as

$$\begin{aligned} X_1\Psi(s) &= \Psi' x, & (X_1X_1)\Psi(s) &= \Psi' + \Psi'' x^2, \\ X_2\Psi(s) &= \Psi' y, & (X_2X_2)\Psi(s) &= \Psi' + \Psi'' y^2, \\ (X_1X_2)\Psi(s) &= \Psi'' xy, \\ (X_2X_1)\Psi(s) &= \Psi'' xy, \\ v_t^\epsilon &= \varepsilon\Psi'. \end{aligned}$$

Let us evaluate the equation on the function v^ε and obtain

$$\begin{aligned} v_t^\varepsilon - \sum_{i,j=1}^{2} &\left(\delta_{ij} - \frac{X_iv^\varepsilon X_jv^\varepsilon}{\varepsilon^2+|\nabla_0 v^\varepsilon|^2}\right)(X_iX_j)v^\varepsilon = \qquad (22) \\ &= \varepsilon\Psi' - \frac{\Big(\varepsilon^2+(X_2v^\varepsilon)^2\Big)(X_1X_1)v^\varepsilon}{\varepsilon^2+|\nabla_0 v^\varepsilon|^2} - \frac{\Big(\varepsilon^2+(X_1v^\varepsilon)^2\Big)(X_2X_2)v^\varepsilon}{\varepsilon^2+|\nabla_0 v^\varepsilon|^2} + 2\frac{X_1v^\varepsilon X_2v^\varepsilon (X_1X_2)v^\varepsilon}{\varepsilon^2+|\nabla_0 v^\varepsilon|^2} \\ &= \varepsilon\Psi' - \frac{\Big(\varepsilon^2+(\Psi'y)^2\Big)(\Psi'+\Psi''x^2)}{\varepsilon^2+|\Psi'x|^2+|\Psi'y|^2} - \frac{\Big(\varepsilon^2+(\Psi'x)^2\Big)(\Psi'+\Psi''y^2)}{\varepsilon^2+|\Psi'x|^2+|\Psi'y|^2} + 2\frac{(\Psi'x)(\Psi'y)\Psi''xy}{\varepsilon^2+|\Psi'x|^2+|\Psi'y|^2}. \end{aligned}$$

We use the fact that $\Psi' \geq 0$ and $|\Psi''| \leq 2\sqrt{3}(\Psi')^{1/2}$ and continue from the last line of (4) as

$$\begin{aligned} &\leq \epsilon\Psi' - \frac{\varepsilon^2\Psi''x^2}{\varepsilon^2+(\Psi')^2(x^2+y^2)} - \frac{\varepsilon^2\Psi''y^2}{\varepsilon^2+(\Psi')^2(x^2+y^2)} \\ &\leq \varepsilon\Psi' + \frac{2\sqrt{3}\,\varepsilon^2(\Psi')^{1/2}(x^2+y^2)}{\varepsilon^2+(\Psi'x)^2+(\Psi'y)^2}. \end{aligned}$$

Recall that $\Psi'=0$ for the case with $(|x|^2+|y|^2)/2 \geq 2$. Thus the estimate for a positive finite fixed number C immediately follows for the trivial case as

$$\varepsilon\Psi' + \frac{2\sqrt{3}\,\varepsilon^2(\Psi')^{1/2}(x^2+y^2)}{\varepsilon^2+(\Psi'x)^2+(\Psi'y)^2} = 0 \leq C\varepsilon^{1/2}.$$

We have two cases for $(|x|^2+|y|^2)/2<2$. The first one is $\varepsilon>\Psi'$ resulting in

$$\varepsilon\Psi' + \frac{2\sqrt{3}\,\varepsilon^2(\Psi')^{1/2}(x^2+y^2)}{\varepsilon^2+(\Psi' x)^2+(\Psi' y)^2} \le \frac{8\sqrt{3}\,\epsilon^{5/2}(x^2+y^2)}{\epsilon^2} \le 32\sqrt{3}\epsilon^{1/2} \le C\varepsilon^{1/2}.$$

In the second case where now $\varepsilon<\Psi'$ we find the same estimate as

$$\varepsilon\Psi' + \frac{2\sqrt{3}\,\varepsilon^2(\Psi')^{1/2}(x^2+y^2)}{\varepsilon^2+(\Psi' x)^2+(\Psi' y)^2} \le \varepsilon\Psi' + \frac{2\sqrt{3}\varepsilon^2(\Psi')^{1/2}(x^2+y^2)}{(\Psi')^2(x^2+y^2)} \le \frac{2\sqrt{3}\varepsilon^2}{(\Psi')^{3/2}} \le \frac{2\sqrt{3}\varepsilon^2}{\epsilon^{3/2}} \le 2\sqrt{3}\varepsilon^{1/2} \le C\epsilon^{1/2}.$$

The rest of the proof follows exactly in the same way as the proof of Theorem 3.1. □

We need the following theorem as well:

Theorem 4.2. *For each $\varepsilon>0$ there exist some finite numbers $B,b>0$ independent of ε such that*

$$\left|1-u^\varepsilon(p,t)\right| \le Be^{-b(x^2+y^2)} \quad \text{for all } (p,t)\in H^1\times[0,T],$$

where T is some positive finite number denoting the final time.

Proof. We may assume that

$$|u_0|\le 1 \text{ on } H^1, \qquad u_0=0 \quad \text{in } \frac{|x|^2+|y|^2}{2}\ge 1. \tag{23}$$

We will estimate the function $|1-u^\varepsilon|$ in terms of the exponential function $\Psi(s)=\hat{c}\mathrm{e}^{-\sigma(2T-\alpha t)s}$ and $v^\epsilon(p,t)=\Psi((x^2+y^2))$ where $0<\alpha<1,\ 0<\sigma<\infty$ and constant $\hat{c}=2\mathrm{e}^{4\sigma T}$.

Now we proceed by using a procedure similar to the one in the proof of [2, Theorem 5.1]. Note that $\Psi'<0$ and $\Psi''>0$. Using the derivatives of v^ε computed in (3.1) the curvature operator on the function v^ε becomes:

$$\begin{aligned}
&v_t^\varepsilon - \sum_{i,j=1}^{2}\left(\delta_{ij}-\frac{X_i v^\varepsilon X_j v^\varepsilon}{\varepsilon^2+|X_j v^\varepsilon|^2}\right)(X_iX_j)v^\varepsilon\\
&= v_t^\varepsilon - \frac{(\varepsilon^2+(\Psi' y)^2)\Psi'+\varepsilon^2\Psi'' x^2}{\varepsilon^2+|\Psi' x|^2+|\Psi' y|^2} - \frac{(\varepsilon^2+(\Psi' x)^2)\Psi'+\varepsilon^2\Psi'' y^2}{\varepsilon^2+|\Psi' x|^2+|\Psi' y|^2}\\
&= \frac{\varepsilon^3\Psi'+\varepsilon(\Psi')^3x^2+\varepsilon(\Psi')^3y^2-\varepsilon^2\Psi'-(\Psi')^3y^2-\varepsilon^2\Psi''x^2-\varepsilon^2\Psi'-(\Psi')^3x^2-\varepsilon^2\Psi''y^2}{\varepsilon^2+|\Psi' x|^2+|\Psi' y|^2}\\
&\ge \frac{(\Psi')^3(x^2+y^2)(\varepsilon-1)+|\Psi'|\varepsilon^2-\Psi''\varepsilon^2(x^2+y^2)}{\varepsilon^2+|\Psi' x|^2+|\Psi' y|^2}\\
&= (\sigma\hat{c}(2T-\alpha t))^3\mathrm{e}^{-3\sigma(2T-\alpha t)(x^2+y^2)}(x^2+y^2)(1-\varepsilon)\\
&\quad + \sigma\hat{c}(2T-\alpha t)\mathrm{e}^{-\sigma(2T-\alpha t)(x^2+y^2)}\varepsilon^2(1-\sigma\hat{c}(2T-\alpha t)(x^2+y^2)) \ge 0,
\end{aligned}$$

for a suitably chosen α and σ.

Note that

$$(1-u^\varepsilon)_t - \sum_{i,j=1}^{2}\left(\delta_{ij}-\frac{X_i(1-u^\varepsilon)X_j(1-u^\varepsilon)}{\varepsilon^2+|\nabla_0(1-u^\varepsilon)|^2}\right)(X_iX_j)(1-u^\varepsilon)=0.$$

Recall that $0\le|1-u_0|\le 2$ due to (4). Hence, applying comparison principle to the functions $1-u^\varepsilon$ (as well as $u^\varepsilon-1$) and v^ε we get

$$|1-u^\varepsilon|\le v^\varepsilon \quad \text{in } H^1\times[0,T].$$

The proof is complete. □

We can now prove our main theorem in the H^1 setting.

Proof. Since now we have both Theorem 4.1 and 4.2 in the Heisenberg setting, Step 1 follows exactly in the same way as in the SE(2) case. So we continue with Step 2.

As in the case of SE(2) we define for $\xi, \eta \in H^1$, $t > 0$ and

$$|\xi - \eta|_0 = |(x_\xi, y_\xi, \theta_\xi) - (x_\eta, y_\eta, \theta)|_{H^1} = |\bar{\xi} - \bar{\eta}|_E = (|x_\xi - x_\eta|^2 + |y_\xi - y_\eta|^2)^{\frac{1}{2}},$$

that

$$\phi(\bar{\xi}, \bar{\eta}, \theta, t) = \mu\epsilon^{1-\frac{\gamma}{2}}|\xi - \eta|_0^\gamma = \mu\epsilon^{1-\frac{\gamma}{2}}|\bar{\xi} - \bar{\eta}|^\gamma = \mu\epsilon^{1-\frac{\gamma}{2}}\big((x_\xi - x_\eta)^2 + (y_\xi - y_\eta)^2\big)^{\gamma/2}.$$

We write the derivatives explicitly as

$$\begin{aligned}
X_1^\xi \phi &= \mu\epsilon^{1-\frac{\gamma}{2}}\gamma(x_\xi - x_\eta)|\bar{\xi} - \bar{\eta}|^{\gamma-2}, \\
X_1^\eta \phi &= -\mu\epsilon^{1-\frac{\gamma}{2}}\gamma(x_\xi - x_\eta)|\bar{\xi} - \bar{\eta}|^{\gamma-2}, \\
X_2^\xi \phi &= \mu\epsilon^{1-\frac{\gamma}{2}}\gamma(y_\xi - y_\eta)|\bar{\xi} - \bar{\eta}|^{\gamma-2}, \\
X_2^\eta \phi &= -\mu\epsilon^{1-\frac{\gamma}{2}}\gamma(y_\xi - y_\eta)|\bar{\xi} - \bar{\eta}|^{\gamma-2}.
\end{aligned}$$

The second order derivatives for $\zeta \in \{\xi, \eta\}$, $i, j \in \{1, 2\}$ and $i \neq j$ are written explicitly as:

$$\begin{aligned}
(X_1^\zeta X_1^\zeta)\phi =& \mu\epsilon^{1-\frac{\gamma}{2}}\gamma|\bar{\xi} - \bar{\eta}|^{\gamma-2} + \mu\epsilon^{1-\frac{\gamma}{2}}\gamma(\gamma-2)(x_\xi - x_\eta)^2|\bar{\xi} - \bar{\eta}|^{\gamma-4}, \\
(X_2^\zeta X_2^\zeta)\phi =& \mu\epsilon^{1-\frac{\gamma}{2}}\gamma|\bar{\xi} - \bar{\eta}|^{\gamma-2} + \mu\epsilon^{1-\frac{\gamma}{2}}\gamma(\gamma-2)(y_\xi - y_\eta)^2|\bar{\xi} - \bar{\eta}|^{\gamma-4}, \\
(X_i^\zeta X_j^\zeta)\phi =& \mu\epsilon^{1-\frac{\gamma}{2}}\gamma(\gamma-2)(x_\xi - x_\eta)(y_\xi - y_\eta)|\bar{\xi} - \bar{\eta}|^{\gamma-4}.
\end{aligned}$$

Since $\sup \omega$ is attained at an interior point $(\hat{\xi}, \hat{\eta}, \hat{\theta}, \hat{t})$ and $u^{\epsilon_1}, u^{\epsilon_2} \in C^\infty$ at the maximum point we have $X_i^\xi u^{\epsilon_1}(\hat{\xi}, \hat{\theta}, \hat{t}) = X_i^\xi \phi(\hat{\xi}, \hat{\eta}, \hat{t})$ and $-X_i^\eta u^{\epsilon_2}(\hat{\eta}, \hat{\theta}, \hat{t}) = X_i^\eta \phi(\hat{\xi}, \hat{\eta}, \hat{t})$ and

$$\begin{aligned}
&\sum_{i,j=1}^{2}\Big(\delta_{ij} - \frac{X_i^\xi u^{\epsilon_1}(\hat{\xi}, \hat{\theta}, \hat{t})X_j^\xi u^{\epsilon_1}(\hat{\xi}, \hat{\theta}, \hat{t})}{\epsilon_1^2 + |\nabla_0^\xi u^{\epsilon_1}(\hat{\xi}, \hat{\theta}, \hat{t})|^2}\Big)\Big((X_i^\xi X_j^\xi)u^{\epsilon_1}(\hat{\xi}, \hat{\theta}, \hat{t}) - (X_i^\xi X_j^\xi)\phi(\hat{\xi}, \hat{\eta}, \hat{t})\Big) \leq 0, \\
&\sum_{i,j=1}^{2}\Big(\delta_{ij} - \frac{X_i^\eta u^{\epsilon_2}(\hat{\eta}, \hat{\theta}, \hat{t})X_j^\xi u^{\epsilon_2}(\hat{\eta}, \hat{\theta}, \hat{t})}{\epsilon_2^2 + |\nabla_\eta^0 u^{\epsilon_2}(\hat{\eta}, \hat{\theta}, \hat{t})|^2}\Big)\Big(-(X_i^\eta X_j^\eta)u^{\epsilon_2}(\hat{\eta}, \hat{\theta}, \hat{t}) - (X_i^\eta X_j^\eta)\phi(\hat{\xi}, \hat{\eta}, \hat{t})\Big) \leq 0.
\end{aligned}$$

Therefore we may write

$$\begin{aligned}
\frac{M\epsilon^\alpha}{2T} =& \partial_t\phi(\hat{\xi}, \hat{\eta}, \hat{t}) = \partial_t u^{\epsilon_1}(\hat{\xi}, \hat{\theta}, \hat{t}) - \partial_t u^{\epsilon_2}(\hat{\eta}, \hat{\theta}, \hat{t}) = \\
=& \sum_{i,j=1}^{2}\Big(\delta_{ij} - \frac{X_i^\xi u^{\epsilon_1}(\hat{\xi}, \hat{\theta}, \hat{t})X_j^\xi u^{\epsilon_1}(\hat{\xi}, \hat{\theta}, \hat{t})}{\epsilon_1^2 + |\nabla_0^\xi u^{\epsilon_1}(\hat{\xi}, \hat{\theta}, \hat{t})|^2}\Big)(X_i^\xi X_j^\xi)u^{\epsilon_1}(\hat{\xi}, \hat{\theta}, \hat{t}) - \\
& \sum_{i,j=1}^{2}\Big(\delta_{ij} - \frac{X_i^\eta u^{\epsilon_2}(\hat{\eta}, \hat{\theta}, \hat{t})X_j^\eta u^{\epsilon_2}(\hat{\eta}, \hat{\theta}, \hat{t})}{\epsilon_2^2 + |\nabla_0^\eta u^{\epsilon_2}(\hat{\eta}, \hat{\theta}, \hat{t})|^2}\Big)(X_i^\eta X_j^\eta)u^{\epsilon_2}(\hat{\eta}, \hat{\theta}, \hat{t}) \leq \\
\leq& \sum_{i,j=1}^{2}\Big(\delta_{ij} - \frac{X_i^\xi \phi(\hat{\xi}, \hat{\eta}, \hat{t})X_j^\xi \phi(\hat{\xi}, \hat{\eta}, \hat{t})}{\epsilon_1^2 + |\nabla_0^\xi \phi(\hat{\xi}, \hat{\eta}, \hat{t})|^2}\Big)(X_i^\xi X_j^\xi)\phi(\hat{\xi}, \hat{\eta}, \hat{t}) - \\
& \sum_{i,j=1}^{2}\Big(\delta_{ij} - \frac{X_i^\eta \phi(\hat{\xi}, \hat{\eta}, \hat{t})X_j^\eta \phi(\hat{\xi}, \hat{\eta}, \hat{t})}{\epsilon_2^2 + |\nabla_0^\eta \phi(\hat{\xi}, \hat{\eta}, \hat{t})|^2}\Big)(X_i^\eta X_j^\eta)\phi(\hat{\xi}, \hat{\eta}, \hat{t}) = \\
=& (\epsilon_1^2 - \epsilon_2^2)\frac{X_i^\xi \phi(\hat{\xi}, \hat{\eta}, \hat{t})X_j^\xi \phi(\hat{\xi}, \hat{\eta}, \hat{t})}{(\epsilon_2^2 + |\nabla_0^\eta \phi(\hat{\xi}, \hat{\eta}, \hat{t})|^2)(\epsilon_1^2 + |\nabla_0^\xi \phi(\hat{\xi}, \hat{\eta}, \hat{t})|^2)}(X_i^\xi X_j^\xi)\phi(\hat{\xi}, \hat{\eta}, \hat{t}) \leq \\
\leq& (\epsilon_1^2 - \epsilon_2^2)\frac{(X_i^\xi X_j^\xi)\phi(\hat{\xi}, \hat{\eta}, \hat{t})}{(\epsilon_1^2 + |\nabla_0^\xi \phi(\hat{\xi}, \hat{\eta}, \hat{t})|^2)} \leq C_1\frac{\mu\gamma^2\epsilon^{3-\frac{\gamma}{2}}|\bar{\xi} - \bar{\eta}|^{\gamma-2}}{\epsilon_1^2 + \mu^2\epsilon^{2-\gamma}\gamma^2|\bar{\xi} - \bar{\eta}|^{2\gamma-2}} \\
=& C_1\gamma^2\mu^{\frac{1}{\gamma-1}}\epsilon^{\frac{\gamma-2}{2(\gamma-1)}}\frac{\epsilon^{\frac{\gamma}{\gamma-1}}\mu^{\frac{\gamma-2}{\gamma-1}}\epsilon^{-\frac{(\gamma-2)^2}{2(\gamma-1)}}|\bar{\xi} - \bar{\eta}|^{\gamma-2}}{\epsilon_1^2 + \mu^2\epsilon(2-\gamma)|\bar{\xi} - \bar{\eta}|^{2\gamma-2}},
\end{aligned}$$

where C_1 denotes a positive finite fixed number. Young's inequality $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$ (where $\frac{1}{p} + \frac{1}{q} = 1$) with $p = \frac{2(\gamma-1)}{\gamma}$ and $q = \frac{2(\gamma-1)}{\gamma-2}$ gives the contradiction for large M values and where C_1, C_2 and C_3 are

some fixed positive finite numbers as follows:

$$\frac{M}{2T}\epsilon^{\alpha} \leq C_1\gamma^2\mu^{\frac{1}{\gamma-1}}\epsilon^{\frac{\gamma-2}{2(\gamma-1)}}\frac{\frac{\gamma}{2(\gamma-1)}\epsilon^2+\frac{\gamma-2}{2(\gamma-1)}\mu^2\epsilon^{2-\gamma}|\bar{\xi}-\bar{\eta}|^{2\gamma-2}}{\epsilon_1^2+\mu^2\epsilon^{2-\gamma}|\bar{\xi}-\bar{\eta}|^{2\gamma-2}}$$
$$\leq C_2\mu^{\frac{1}{\gamma-1}}\epsilon^{\frac{\gamma-2}{2(\gamma-1)}} = C_2\Big(\frac{\gamma 4^{\gamma}\mathrm{Lip}(u_0)^{\gamma}}{M^{\gamma-1}}\Big)^{\frac{1}{\gamma-1}}\epsilon^{\alpha} = C_3\frac{\epsilon^{\alpha}}{M},$$

where instead of (3.2) here we used $\gamma = \frac{2(1-\alpha)}{1-2\alpha} > 2$ in order to define $\gamma \in \mathbb{R}$.

□

Remark. *We remark that Theorem 1.1 and Corollary 1.1.1 hold also when vanishing viscosity solution u is spatially periodic over $G = SE(2)$ or H^1.*

References

[1] Altschuler, S. J., M. A. Grayson, et al. (1991). *Shortening space curves and flow through singularities.* Institute for Mathematics and its Applications (USA).

[2] Bonfiglioli, A., E. Lanconelli, F. Uguzzoni, et al. (2002). Uniform gaussian estimates for the fundamental solutions for heat operators on carnot groups. *Advances in Differential equations 7*(10), 1153–1192.

[3] Brakke, K. (1978). The motion ofa surface by its mean curvature. *Mathematical Notes Series, Princeton University Press, Princeton, NJ*.

[4] Brakke, K. A. (2015). *The Motion of a Surface by Its Mean Curvature.(MN-20).* Princeton University Press.

[5] Capogna, L. and G. Citti (2009). Generalized mean curvature flow in carnot groups. *Communications in Partial Differential Equations 34*(8), 937–956.

[6] Capogna, L., D. Danielli, S. D. Pauls, and J. Tyson (2007). *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, Volume 259. Springer Science & Business Media.

[7] Chen, Y. G., Y. Giga, S. Goto, et al. (1991). Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom 33*(3), 749–786.

[8] Cheng, J.-H., J.-F. Hwang, A. Malchiodi, and P. Yang (2005). Minimal surfaces in pseudohermitian geometry. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze 4*(1), 129–177.

[9] Citti, G., B. Franceschiello, G. Sanguinetti, and A. Sarti (2015). Sub-riemannian mean curvature flow for image processing. *arXiv preprint arXiv:1504.03710*.

[10] Citti, G. and A. Sarti (2006). A cortical based model of perceptual completion in the roto-translation space. *Journal of Mathematical Imaging and Vision 24*(3), 307–326.

[11] Crandall, M. G., H. Ishii, and P.-L. Lions (1992). User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society 27*(1), 1–67.

[12] Danielli, D., N. Garofalo, and D.-M. Nhieu (2007). Sub-riemannian calculus on hypersurfaces in carnot groups. *Advances in Mathematics 215*(1), 292–378.

[13] Deckelnick, K. (2000). Error bounds for a difference scheme approximating viscosity solutions of mean curvature flow. *Interfaces and Free Boundaries 2*(2), 117–142.

[14] Dirr, N., F. Dragoni, and M. von Renesse (2010). Evolution by mean curvature flow in sub-riemannian geometries. *Communications on Pure and Applied Mathematics 9*(2), 307–326.

[15] Evans, L. C. and J. Spruck (1992a). Motion of level sets by mean curvature. ii. *Transactions of the american mathematical society 330*(1), 321–332.

[16] Evans, L. C. and J. Spruck (1992b). Motion of level sets by mean curvature iii. *The Journal of Geometric Analysis 2*(2), 121–150.

[17] Evans, L. C. and J. Spruck (1995). Motion of level sets by mean curvature iv. *The Journal of Geometric Analysis 5*(1), 77–114.

[18] Evans, L. C., J. Spruck, et al. (1991). Motion of level sets by mean curvature i. *J. Diff. Geom 33*(3), 635–681.

[19] Ferrari, F., Q. Liu, and J. J. Manfredi (2014). On the horizontal mean curvature flow for axisymmetric surfaces in the heisenberg group. *Communications in Contemporary Mathematics 16*(03), 1350027.

[20] Gage, M. et al. (1983). An isoperimetric inequality with applications to curve shortening. *Duke Math. J 50*(4), 1225–1229.

[21] Gage, M. and R. S. Hamilton (1986). The heat equation shrinking convex plane curves. *Journal of Differential Geometry 23*(1), 69–96.

[22] Gage, M. E. (1984). Curve shortening makes convex curves circular. *Inventiones mathematicae 76*(2), 357–364.

[23] Grayson, M. A. (1987). The heat equation shrinks embedded plane curves to round points. *Journal of Differential geometry 26*(2), 285–314.

[24] Hladky, R. K., S. D. Pauls, et al. (2008). Constant mean curvature surfaces in sub-riemannian geometry. *Journal of Differential Geometry 79*(1), 111–139.

[25] Huisken, G. et al. (1984). *Flow by mean curvature of convex surfaces into spheres.* Australian National University, Centre for Mathematical Analysis.

[26] Merriman, B., J. K. Bence, and S. Osher (1992). *Diffusion generated motion by mean curvature.* Department of Mathematics, University of California, Los Angeles.

[27] Nagel, A., E. M. Stein, and S. Wainger (1985). Balls and metrics defined by vector fields i: Basic properties. *Acta Mathematica 155*(1), 103–147.

[28] Petitot, J. and Y. Tondut (1999). Vers une neurogéométrie. fibrations corticales, structures de contact et contours subjectifs modaux. *Mathématiques informatique et sciences humaines* (145), 5–102.